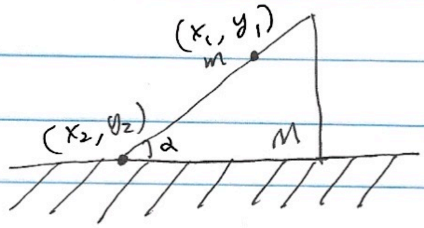
Goldstein 2.20

We give the system 4 coordinates: (x_1, y_1), (x_2, y_2), correspond to location of the particle and location of the tip of wedge.



Evidently, the Lagrangian is

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}M(\dot{x}_2^2 + \dot{y}_2^2) - mgy_1 - Mgy_2$$

giving

$$\begin{cases} m\ddot{x}_1 = Q_{x_1} \\ m\ddot{y}_1 + mg = Q_{y_1} \\ M\ddot{x}_2 = Q_{x_2} \\ M\ddot{y}_2 + Mg = Q_{y_2} \end{cases}$$

where $Q_{x_1}, Q_{y_1}, Q_{x_2}, Q_{y_2}$ are forces of constraints for the corresponding variables to be solved.

There are two constraint eq.

$$y_2 = 0, \quad \frac{(y_1 - y_2)}{(x_1 - x_2)} = \tan\alpha.$$

$$\begin{cases} f_1 = y_1 - (x_1 - x_2)\tan\alpha = 0 \\ f_2 = y_2 = 0. \end{cases}$$

Goldstein (eq 2.25) Using the formula for the force of constraint, we have.

$$\begin{cases} Q_{x_1} = -\lambda_1 \tan\alpha \\ Q_{y_1} = \lambda_1 \\ Q_{x_2} = \lambda_1 \tan\alpha \\ Q_{y_2} = \lambda_2 \end{cases}$$

Putting it with the Lagrangian derived equations, we have

$$\begin{cases} m\ddot{x}_1 = -\lambda_1 \tan\alpha \\ m\ddot{y}_1 + mg = \lambda_1 \\ M\ddot{x}_2 = \lambda_1 \tan\alpha \\ M\ddot{y}_2 + Mg = \lambda_2 \\ y_1 - (x_1 - x_2)\tan\alpha = 0 \\ y_2 = 0 \end{cases}$$

6 equations for 6 variables, so can be solved.

$$y_2 = 0 \Rightarrow \ddot{y}_2 = 0 \Rightarrow \boxed{\lambda_2 = Mg}$$

$$m y_1 = m(x_1 - x_2)\tan\alpha, \quad m\ddot{y}_1 = m(\ddot{x}_1 - \ddot{x}_2)\tan\alpha$$

$$m(\ddot{x}_1 - \ddot{x}_2)\tan\alpha + mg = \lambda_1$$

$$m\tan\alpha\left(-\frac{\lambda_1}{m}\tan\alpha - \frac{\lambda_1}{M}\tan\alpha\right) + mg = \lambda_1$$

$$mg = \lambda_1\left[1 + \left(1 + \frac{m}{M}\right)\tan^2\alpha\right].$$

$$\lambda_1 = \frac{mg}{\left[1 + \left(1 + \frac{m}{M}\right)\frac{\sin^2\alpha}{\cos^2\alpha}\right]}$$

$$= \boxed{\frac{mg\cos^2\alpha}{1 + \frac{m}{M}\sin^2\alpha}}$$

These completes the solution for the forces of constraint.

$$\begin{cases} \lambda_2 = Mg \\ \lambda_1 = mg\dfrac{\cos^2\alpha}{1 + \frac{m}{M}\sin^2\alpha} = mg\dfrac{1}{1 + \left(1 + \frac{m}{M}\right)\tan^2\alpha} \end{cases}$$

It's then straightforward to solve for x_1, y_1, x_2:

$$x_1 = -\frac{\lambda_1}{m}\tan\alpha \frac{1}{2}t^2$$
$$y_1 = \left(\frac{\lambda_1}{m} - g\right)\frac{1}{2}t^2$$
$$x_2 = \frac{\lambda_1}{M}\tan\alpha \frac{1}{2}t^2$$

Now compute work done by forces of constraint:

$\vec{V}$ stands for velocity, since we are interested in direction of $\vec{x}$.

$$\cancel{\vec{x}_m}\ \vec{V}_m \propto \left(-\frac{\lambda_1}{m}\tan\alpha,\ \left(\frac{\lambda_1}{m} - g\right)\right) \qquad \vec{V}_M \propto \left(\frac{\lambda_1}{M}\tan\alpha,\ 0\right)$$

$$\vec{F}_m^c = (-\lambda_1\tan\alpha,\ \lambda_1) \qquad F_M^c = (\lambda_1\tan\alpha,\ \lambda_2)$$

$$W \propto \vec{V}_m \cdot \vec{F}_m^c + \vec{V}_M \cdot F_M^c \qquad \left(dW = \vec{F}\cdot d\vec{x} = \vec{F}\cdot\frac{d\vec{x}}{dt}\,dt\right).$$

$$= \frac{\lambda_1^2}{m}\tan^2\alpha + \frac{\lambda_1^2}{m} - g\lambda_1 + \frac{\lambda_1^2}{M}\tan^2$$

$$= 0$$

The forces of constraint do no net work, as it should.

From the eqm: $m\ddot{x}_1 = -\lambda_1\tan\alpha$, $M\ddot{x}_2 = \lambda_1\tan\alpha$ we can obtain

$$\frac{d}{dt}(m\dot{x}_1 + M\dot{x}_2) = 0,$$ that is, the linear momentum is conserved.